

Consistency relations and conservation of ζ in holographic inflation

Jaume Garriga^{a,b}, Yuko Urakawa^c

a. Departament de Física Fonamental i Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain

b. Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, MA 02155, USA

c. Department of Physics and Astrophysics, Nagoya University, Chikusa, Nagoya 464-8602, Japan

ABSTRACT: It is well known that, in single clock inflation, the curvature perturbation ζ is constant in time on superhorizon scales. In the standard bulk description this follows quite simply from the local conservation of the energy momentum tensor in the bulk. On the other hand, in a holographic description, the constancy of the curvature perturbation must be related to the properties of the RG flow in the boundary theory. Here, we show that, in single clock holographic inflation, the time independence of correlators of ζ follows from the cut-off independence of correlators of the energy momentum tensor in the boundary theory, and from the so-called consistency relations for vertex functions with a soft leg.

KEYWORDS: Inflation, dS/CFT correspondence, Primordial perturbation.

Contents

1. Introduction	1
2. Wave function prescription	2
2.1 Wave functional	2
2.2 Tree level ζ correlators from the wave function	3
3. Consistency relations from diffeomorphism invariance	5
3.1 Dilatation invariance of the wave function	5
3.2 Ward-Takahashi identity	6
3.3 Consistency relation for vertices with one soft leg	7
4. ζ correlators from holography	8
4.1 Holographic prescription	8
4.2 Vertex functions	9
5. Conservation of ζ and consistency relation	10
5.1 Cutoff independence of the energy momentum tensor	10
5.2 Consistency relation and coincidence limit	11
5.3 Conservation of ζ	12
5.4 Primordial spectra from holographic inflation	13
5.5 Relation to previous work	13
6. Condition on boundary theory from diffeomorphism invariance	14
7. Conclusions	16
A. Consistency relations for ζ correlators	17

1. Introduction

In a holographic description of inflation, the renormalization scale μ in the boundary theory is expected to correspond to a temporal coordinate in the bulk. In a recent paper [1] we investigated the issue of time evolution of the curvature perturbation ζ by considering a generic deformed CFT at the boundary, in the limit where conformal perturbation theory is valid along the RG flow between two nearby fixed points. We concluded that the two point function for ζ is conserved along the RG flow provided that we make the identification¹

$$\mu \propto a,$$

¹This relation may have slow roll corrections, beyond the leading order in conformal perturbation theory which was considered in Ref. [1].

where a is the cosmological scale factor.

On the other hand, in Ref. [1] we were not able to show the conservation of higher order correlators of ζ . This is technically complicated, due to the presence of semi-local terms in the relation between higher order cosmological correlators and boundary correlators. The renormalization of expressions containing such semi-local terms (where two or more of the points in the correlator coincide) is not straightforward, and it is hard to check explicitly whether these expressions depend on the renormalization scale. To overcome this difficulty, here we will use a different approach, which does not rely on conformal perturbation theory.

Our strategy will be based on the observation that, in a renormalizable quantum field theory, there is an energy momentum tensor whose correlators do not depend on the renormalization scale [2]. We will also use the so-called consistency relations, which express higher order vertices with a soft leg in terms of lower order vertices. In this way, we can address the conservation of n -point correlators of ζ recursively, starting with the 2-point function.

The paper is organized as follows. In Section 2, we discuss our setup and conventions. Section 3 deals with the consistency relations involving correlators with soft legs. In Section 4 we express the correlators of ζ in terms of correlators of the energy momentum tensor in the boundary theory. In Section 5 we discuss the conditions which are necessary for the conservation of ζ from the point of view of the boundary theory. In Section 6 we generalize our arguments to the case of tensor perturbations. Our conclusions are summarized in Section 7.

2. Wave function prescription

Correlation functions of primordial perturbations can be obtained from the cosmological wave function [3, 4, 5]. In holography, the wave function of long wavelength modes is related to the generating functional of the boundary quantum field theory (QFT). For the moment, however, we will not assume the holographic relation and we will simply discuss the correlation functions obtained from a given wave function.

2.1 Wave functional

We consider a wave function on a particular time slicing Σ_t such that the gauge condition $\delta\phi = 0$ is satisfied. With an appropriate choice of spatial coordinates, we express the d -dimensional spatial line element as

$$dl_d^2 = a^2(t) e^{2\zeta(t, \mathbf{x})} d\mathbf{x}^2. \quad (2.1)$$

Here, we neglected the tensor perturbation, which will be discussed in Sec. 6. We assume that the wave function of the $(d + 1)$ dimensional bulk spacetime is given by a functional of the curvature perturbation ζ on the slicing $\delta\phi = 0$,

$$\psi_t = \psi_t[\zeta(t, \mathbf{x})], \quad (2.2)$$

which will become a good approximation in case the universe is dominated by a single scalar degree of freedom.

The probability distribution function is given by

$$P_t[\zeta] = |\psi_t[\zeta]|^2 \equiv e^{-W_t[\zeta]}, \quad (2.3)$$

and satisfies the normalization condition

$$\int D\zeta P_t[\zeta] = 1. \quad (2.4)$$

The n -point functions for ζ on Σ_t can then be obtained as

$$\langle \zeta(\mathbf{x}_1) \zeta(\mathbf{x}_2) \cdots \zeta(\mathbf{x}_n) \rangle = \int D\zeta P_t[\zeta] \zeta(\mathbf{x}_1) \zeta(\mathbf{x}_2) \cdots \zeta(\mathbf{x}_n). \quad (2.5)$$

In single clock inflation, where only the adiabatic mode is relevant, ζ becomes time independent on superhorizon scales. In single field models this happens once the decaying mode becomes negligibly small.

For later use, we expand $W_t[\zeta] = -\ln P_t[\zeta]$ as

$$W_t[\zeta] = \sum_{n=2}^{\infty} \frac{1}{n!} \int d^d \mathbf{x}_1 \cdots \int d^d \mathbf{x}_n W^{(n)}(t; \mathbf{x}_1, \cdots, \mathbf{x}_n) \zeta(\mathbf{x}_1) \cdots \zeta(\mathbf{x}_n), \quad (2.6)$$

where we introduced the vertex functions

$$W^{(n)}(t; \mathbf{x}_1, \cdots, \mathbf{x}_n) \equiv \left. \frac{\delta^n W_t[\zeta]}{\delta \zeta(\mathbf{x}_1) \cdots \delta \zeta(\mathbf{x}_n)} \right|_{\zeta=0}. \quad (2.7)$$

As we shall see in Section 3, the tadpole term with $W^{(1)}$ is required to vanish by Diff invariance.

2.2 Tree level ζ correlators from the wave function

Assuming that the amplitude of ζ is perturbatively small, the n -point function of ζ can be given in terms of $W^{(m)}$ with $m \leq n$. A more detailed discussion of this perturbative expansion can be found in Ref. [4]. For later use, here we reproduce the tree level expressions for the lowest order correlators.

The power spectrum of ζ is given by using the inverse matrix of $W^{(2)}(\mathbf{x}_1, \mathbf{x}_2)$ as

$$\langle \zeta(\mathbf{x}_1) \zeta(\mathbf{x}_2) \rangle = W^{(2)-1}(\mathbf{x}_1, \mathbf{x}_2). \quad (2.8)$$

We assume invariance under global translations and rotations. Then, we can express $W^{(n)}$ in Fourier space as

$$(2\pi)^d \delta \left(\sum_{i=1}^n \mathbf{k}_i \right) \hat{W}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) \equiv \prod_{i=1}^n \int d^d \mathbf{x}_i e^{-i\mathbf{k}_i \cdot \mathbf{x}_i} W^{(n)}(\mathbf{x}_1, \cdots, \mathbf{x}_n). \quad (2.9)$$

Using the Fourier mode $\hat{W}^{(2)}(k)$ with $k \equiv |\mathbf{k}|$, the power spectrum of the curvature perturbation is given by

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \rangle = (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2) P(k_1) \quad (2.10)$$

with

$$P(k) = \frac{1}{\hat{W}^{(2)}(k)}. \quad (2.11)$$

The bi-spectrum for $\zeta(\mathbf{x})$ is expressed by the cubic interaction $W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ as

$$\langle \zeta(\mathbf{x}_1) \zeta(\mathbf{x}_2) \zeta(\mathbf{x}_3) \rangle = - \int \prod_{i=1}^3 d^d \mathbf{y}_i W^{(2)-1}(\mathbf{x}_i, \mathbf{y}_i) W^{(3)}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3), \quad (2.12)$$

Here, we need the minus sign, since the three-point vertex is given by $-W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. In Fourier space, we have

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle_{\text{conn}} = (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(k_1, k_2, k_3) \quad (2.13)$$

with

$$B(k_1, k_2, k_3) = - \frac{\hat{W}^{(3)}(k_1, k_2, k_3)}{\hat{W}^{(2)}(k_1) \hat{W}^{(2)}(k_2) \hat{W}^{(2)}(k_3)} = - \hat{W}^{(3)}(k_1, k_2, k_3) \prod_{i=1}^3 P(k_i). \quad (2.14)$$

The tri-spectrum is composed of the two-different diagrams (see Fig. 2 of Ref. [4]). In Fourier space, it is given by

$$\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \zeta(\mathbf{k}_4) \rangle_{\text{conn}} = (2\pi)^d \delta \left(\sum_{i=1}^4 \mathbf{k}_i \right) T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4), \quad (2.15)$$

with

$$T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) + T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \\ + T_2(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_4) + T_2(\mathbf{k}_1, \mathbf{k}_4, \mathbf{k}_3, \mathbf{k}_1), \quad (2.16)$$

$$T_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = - \hat{W}^{(4)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \prod_{i=1}^4 P(k_i), \quad (2.17)$$

$$T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \hat{W}^{(3)}(k_1, k_2, k_{12}) \hat{W}^{(3)}(k_3, k_4, k_{34}) P(k_{12}) \prod_{i=1}^4 P(k_i), \quad (2.18)$$

where we introduced the momentum \mathbf{k}_{ij} and its absolute value as $\mathbf{k}_{ij} \equiv \mathbf{k}_i + \mathbf{k}_j$ and $k_{ij} \equiv |\mathbf{k}_{ij}|$. Note that using the bi-spectrum $B(k_1, k_2, k_3)$, we can express $T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ as

$$T_2(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \frac{B(k_1, k_2, k_{12}) B(k_3, k_4, k_{34})}{P(k_{12})}. \quad (2.19)$$

Similarly, we can express the n -point function of ζ , using $\hat{W}^{(m)}$ with $m \leq n$.

3. Consistency relations from diffeomorphism invariance

In this section, we derive a Ward-Takahashi identity from diffeomorphism invariance (see also Refs. [5, 6] for a related relevant discussion.)

Note that the wave function characterizes the bulk spacetime beyond the tree level perturbative analysis, and can be used in order to compute correlators to any loop order. Here, imposing diffeomorphism invariance on the $\delta\phi = 0$ slicing, we derive a condition on $W^{(n)}$. When the amplitude of ζ is sufficiently small, we can perturbatively compute the correlators of ζ [4] from the vertex functions $W^{(n)}$, as discussed in Section 2. At the tree level, the condition on $W^{(n)}$ leads to the standard consistency relation for correlators of ζ . It should be stressed, however, that the condition on $W^{(n)}$ holds non-perturbatively, and is therefore more fundamental.

3.1 Dilatation invariance of the wave function

Among the coordinate transformations, we consider the dilatation

$$\mathbf{x} \rightarrow \mathbf{x}_s \equiv e^s \mathbf{x} \quad (3.1)$$

with a constant parameter s , under which the spatial line element, given in Eq. (2.1), is rewritten as

$$\frac{dl_d^2}{a^2(t)} = e^{2\zeta(t, \mathbf{x})} d\mathbf{x}^2 = e^{2\zeta_s(t, \mathbf{x}_s)} d\mathbf{x}_s^2 = e^{2\{\zeta_s(t, e^s \mathbf{x}) + s\}} d\mathbf{x}^2. \quad (3.2)$$

Then, under the dilatation, the curvature perturbation is changed into

$$\zeta_s(t, \mathbf{x}) = \zeta(t, e^{-s} \mathbf{x}) - s. \quad (3.3)$$

The change of the coordinates $\Delta \mathbf{x} \equiv \mathbf{x}_s - \mathbf{x}$ increases with distance to the origin. However, the change $\Delta \mathbf{x}$ can stay perturbatively small in the observable region, which is necessarily bounded. Similar (residual) gauge transformations which diverge in the limit $\mathbf{x} \rightarrow \infty$ can be found in other gauge theories, such as QED, and it is known that soft theorems can be derived by using such residual gauge transformations (see, e.g., Refs. [7, 8, 9]).

The diffeomorphism invariance of the wave function requires that the probability distribution function $P_t[\zeta] = e^{-W_t[\zeta]}$ should be invariant under the dilatation²

$$W_t[\zeta(\mathbf{x})] = W_t[\zeta(e^{-s} \mathbf{x}) - s]. \quad (3.4)$$

²Correlation functions computed by using a Diff invariant probability distribution and measure of integration are, strictly speaking, ill defined. To make them well defined we must factor out the infinite volume of the orbits of the gauge group. In this case, s is the additive parameter in the dilatation group, and so we need to factor out $\int ds$. This is easily achieved by the standard Fadeev-Popov (FP) trick of introducing a Gaussian factor $\exp\{-(G[\zeta])^2/\alpha\}$ in the integrand, accompanied by the determinant $|\partial G[\zeta]/\partial s|$. Here $\alpha > 0$ is an arbitrary positive constant, and $G[\zeta]$ is an s dependent function. A convenient choice for G is the average value of the curvature perturbation in the region of our interest, $G[\zeta] \equiv \bar{\zeta} = (\int d^3x \zeta)/(\int d^3x)$. Under gauge transformation, we have $\bar{\zeta} \rightarrow \bar{\zeta} - s$. In this case, the determinant is constant, and there is no need to introduce FP ghosts. In summary, the Diff invariant exponent in the distribution function of the functional integrand, $W = -\ln P$, gets replaced by $\tilde{W} = W + \bar{\zeta}^2/\alpha$. It is straightforward to check that this modification does not change the correlation functions, since (in the limit of infinite volume) the second term in \tilde{W} has vanishing functional derivative with respect to the curvature perturbation.

In Sec. 6, this argument will be briefly extended to include the tensor perturbation³, and we will comment on its implications for the boundary theory.

3.2 Ward-Takahashi identity

Next, we derive the Ward-Takahashi identity associated with the dilatation invariance. The curvature perturbation after the scale transformation is given by

$$\begin{aligned}\zeta(e^{-s}\mathbf{x}) - s &= \sum_{n=0}^{\infty} \frac{s^n}{n!} \frac{d^n}{ds^n} \zeta(e^{-s}\mathbf{x}) \Big|_{s=0} - s \\ &= \zeta(\mathbf{x}) - s(\mathbf{x} \cdot \partial_{\mathbf{x}} \zeta(\mathbf{x}) + 1) + \sum_{s=2}^{\infty} \frac{(-s)^n}{n!} (\mathbf{x} \cdot \partial_{\mathbf{x}})^n \zeta(\mathbf{x}),\end{aligned}\quad (3.5)$$

where on the second equality, we replaced d/ds with $-\mathbf{x} \cdot \partial_{\mathbf{x}}$. Using Eq. (3.5), we find that at $\mathcal{O}(s)$, Eq. (3.4) gives

$$\mathcal{O}(s): \quad 0 = \int d^d \mathbf{x} \frac{\delta W_t[\zeta]}{\delta \zeta(\mathbf{x})} \{1 + \mathbf{x} \cdot \partial_{\mathbf{x}} \zeta(\mathbf{x})\}, \quad (3.6)$$

and at $\mathcal{O}(s^2)$, it gives

$$\begin{aligned}\mathcal{O}(s^2): \quad 0 &= \int d^d \mathbf{x} \frac{\delta W_t[\zeta]}{\delta \zeta(\mathbf{x})} (\mathbf{x} \cdot \partial_{\mathbf{x}})^2 \zeta(\mathbf{x}) \\ &\quad + \int d^d \mathbf{x}_1 \int d^d \mathbf{x}_2 \frac{\delta^2 W_t[\zeta]}{\delta \zeta(\mathbf{x}_1) \delta \zeta(\mathbf{x}_2)} \{1 + \mathbf{x}_1 \cdot \partial_{\mathbf{x}_1} \zeta(\mathbf{x}_1)\} \{1 + \mathbf{x}_2 \cdot \partial_{\mathbf{x}_2} \zeta(\mathbf{x}_2)\},\end{aligned}\quad (3.7)$$

and so on.

It follows from (3.6) with $\zeta(\mathbf{x}) = 0$ that $\int d^d \mathbf{x} W^{(1)} = 0$. By translation invariance, this implies

$$W^{(1)} = 0. \quad (3.8)$$

In particular, the tadpole term in (2.6) will not contribute, even if the average value of the curvature perturbation $\bar{\zeta}$ is non-vanishing. Likewise, from (3.7) with $\zeta(\mathbf{x}) = 0$, we find $\int d^d \mathbf{x}_1 d^d \mathbf{x}_2 W^{(2)}(t; \mathbf{x}_1, \mathbf{x}_2) = 0$. It follows that, in momentum space,

$$\hat{W}^{(2)}(0, 0) = 0. \quad (3.9)$$

This means that the tree level dispersion of $\hat{\zeta}(\mathbf{k} = 0)$ will be infinite⁴.

³Here, we impose the diffeomorphism invariance on the d -dimensional time slicing Σ_t . When we keep only the d -dimensional diffeomorphism invariance, but we break the $(d+1)$ -dimensional diffeomorphism invariance as in Horava-Lifshitz theory, typically there appears an additional scalar degree of freedom. This case is excluded in our setup where the wave function ψ_t is expressed only by the single (scalar) degree of freedom.

⁴The FP Gaussian $e^{-\bar{\zeta}^2/\alpha}$ discussed in footnote 2 will make the dispersion of $\bar{\zeta}$ finite, but the dispersion of $\hat{\zeta}(\mathbf{k} = 0)$ will still be infinite, since both variables are related by an infinite volume factor. Note that $\hat{\zeta}(\mathbf{k}) = \bar{\zeta} \delta(\mathbf{k} = 0) + \dots$, where the ellipsis denote contributions with $\mathbf{k} \neq 0$.

3.3 Consistency relation for vertices with one soft leg

Taking $(n-1)$ -derivatives with respect to $\zeta(\mathbf{x}_i)$ for $i = 1, \dots, n-1$ on Eq. (3.6), we obtain

$$0 = \int d^d \mathbf{x} \left[\{1 + \mathbf{x} \cdot \partial_{\mathbf{x}} \zeta(\mathbf{x})\} \frac{\delta^n W_t[\zeta]}{\delta \zeta(\mathbf{x}) \delta \zeta(\mathbf{x}_1) \cdots \delta \zeta(\mathbf{x}_{n-1})} + \sum_{i=1}^{n-1} \mathbf{x} \cdot \partial_{\mathbf{x}} \delta(\mathbf{x} - \mathbf{x}_i) \frac{\delta^{n-1} W_t[\zeta]}{\delta \zeta(\mathbf{x}) \cdots \delta \zeta(\mathbf{x}_{i-1}) \delta \zeta(\mathbf{x}_{i+1}) \cdots \delta \zeta(\mathbf{x}_{n-1})} \right]. \quad (3.10)$$

Notice that the functional derivative in the second term excludes the derivative with respect to $\zeta(\mathbf{x}_i)$. Performing the integration by parts and setting $\zeta = 0$, we obtain

$$\int d^d \mathbf{x} W^{(n)}(t; \mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_{n-1}) - \sum_{i=1}^{n-1} \partial_{\mathbf{x}_i} \left\{ \mathbf{x}_i W^{(n-1)}(t; \mathbf{x}_1, \dots, \mathbf{x}_{n-1}) \right\} = 0. \quad (3.11)$$

This is the generalized consistency relation for ζ , which does not require the validity of the perturbative analysis.

It may be convenient to express Eq. (3.11) in Fourier space. Multiplying Eq. (3.11) by $\prod_{i=1}^n \int d^d \mathbf{x}_i e^{-i \mathbf{k}_i \cdot \mathbf{x}_i}$, and using Eq. (2.9), we express Eq. (3.11) as

$$0 = \hat{W}^{(n)}(t; \mathbf{k} = 0, \{\mathbf{k}_i\}_{n-1}) + \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} - d \right) \hat{W}^{(n-1)}(t; \{\mathbf{k}_i\}_{n-1}), \quad (3.12)$$

where we used

$$\begin{aligned} & \sum_{i=1}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} \left[\delta \left(\sum_{i=1}^{n-1} \mathbf{k}_i \right) \hat{W}^{(n-1)}(t; \{\mathbf{k}_i\}_n) \right] \\ &= \delta \left(\sum_{i=1}^{n-1} \mathbf{k}_i \right) \times \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} - d \right) \hat{W}^{(n-1)}(t; -\mathbf{K}_{2,n-1}, \mathbf{k}_2, \dots, \mathbf{k}_{n-1}), \end{aligned} \quad (3.13)$$

and removed a delta function which appears as a common factor in the two terms. Here, $\{\mathbf{k}_i\}$ denotes $(n-1)$ momenta \mathbf{k}_i with $i = 1, \dots, n-1$ which satisfies $\sum_{i=1}^{n-1} \mathbf{k}_i = 0$ and in the second line, we replaced \mathbf{k}_1 with $\mathbf{K}_{m,n}$ defined as

$$\mathbf{K}_{m,n} \equiv \sum_{i=m}^n \mathbf{k}_i. \quad (3.14)$$

Equation (3.13) can be verified by operating $\int d^d \mathbf{k}_1$ on the both sides. Notice that Eq. (3.12) states that if $\hat{W}^{(n-1)}$ does not depend on time, neither does $\hat{W}^{(n)}$ with one soft leg.

When the amplitude of ζ is perturbatively small, as discussed in Appendix A, Eq. (3.12) simply yields the consistency relation in a $(d+1)$ dimensional spacetime, given by

$$\lim_{k_n \rightarrow 0} \frac{\mathcal{C}^{(n)}(\{\mathbf{k}_i\}_n)}{P(k_n)} = - \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} + d(n-2) \right) \mathcal{C}^{(n-1)}(\{\mathbf{k}_i\}_{n-1}), \quad (3.15)$$

where $\mathcal{C}^{(n)}$ denotes the n -point function of ζ with the momentum conservation factor

$$(2\pi)^d \delta \left(\sum_{i=1}^n \mathbf{k}_i \right)$$

removed. In Ref. [10], the consistency relation (3.15) was derived for $d = 3$. The argument in Appendix A shows that an extension to a general spacetime dimension proceeds straightforwardly. The consistency relation involves a soft mode which is induced by a coordinate transformation. Therefore, we do not expect any influence of such soft mode in correlators of a variable which remains invariant under the dilatation. This was explicitly shown in Refs. [11, 12].

In rewriting the WT identity (3.12) in the form (3.15), we implicitly assume the continuity of $\hat{W}^{(3)}$ at $\mathbf{k} = 0$. It has been argued in Refs. [6, 13] that this follows from the constancy of ζ at long wavelengths (see also Ref. [14]). For the purposes of this paper, we will use the consistency relation in the form (3.12). In the holographic context, its continuity at $\mathbf{k} = 0$ requires separate justification. We will come back to this issue in Section 5.

From Eq. (3.7), we can perturbatively derive the consistency relation which connects the n -point function of ζ with two soft legs to the m -point functions with $m < n$. This also can be derived by sending another momentum to 0 in the consistency relation (3.12) with one soft leg.

4. ζ correlators from holography

Our previous discussion is based on the use of the wave function for single field inflationary models. Our next task is to introduce the holographic duality between the $(d + 1)$ -dimensional cosmological spacetime and the d -dimensional field theory at the boundary. The gauge/gravity duality in the inflationary setup is discussed, e.g., in Refs. [3, 4, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

4.1 Holographic prescription

Following Ref. [46], we will assume that the probability distribution of the bulk gravitational field is related to the generating functional of a boundary QFT as

$$P[h, \phi] = |\psi_{\text{bulk}}[h, \phi]|^2 \propto Z_{\text{QFT}}[h, \phi] Z_{\text{QFT}}^*[h, \phi], \quad (4.1)$$

where the generating functional Z_{QFT} is given by

$$Z_{\text{QFT}}[h, \phi] = e^{-W_{\text{QFT}}[h, \phi]} = \int D\chi \exp(-S_{\text{QFT}}[\chi, h, \phi]). \quad (4.2)$$

Here, χ stands for the set of boundary fields. In the boundary, the path integral is doubled by multiplying the generating functional and its complex conjugate together. Comparing

Eq. (4.1) to Eq. (2.3), we find that $W^{(n)}$ are given by

$$W^{(n)}(t(\mu); \mathbf{x}_1, \dots, \mathbf{x}_n) = 2\text{Re} \left[\frac{\delta^n W_{\text{QFT}}[\zeta]}{\delta\zeta(\mathbf{x}_1) \cdots \delta\zeta(\mathbf{x}_n)} \Big|_{\zeta=0} \right]. \quad (4.3)$$

Here, we expressed the time dependence in terms of the renormalization scale μ , postulating that the time evolution of the bulk spacetime is described by the renormalization group flow.

4.2 Vertex functions

We may now express $W^{(n)}$ in terms of boundary correlators of the energy-momentum tensor, defined by

$$T_{ij} \equiv -\frac{2}{\sqrt{h}} \frac{\delta S_{\text{QFT}}}{\delta h^{ij}}. \quad (4.4)$$

The derivative of the boundary action with respect to ζ is then given by the trace part of the energy-momentum tensor, as

$$\frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x})} = e^{(d-2)\zeta(\mathbf{x})} \delta^{ij} T_{ij}[\zeta](\mathbf{x}). \quad (4.5)$$

Using Eq. (4.5), we easily find

$$W^{(2)}(\mathbf{x}_1, \mathbf{x}_2) = -2\text{Re} \left[\left\langle \frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_1)} \frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_2)} \right\rangle \Big|_{\zeta=0} \right] = -2\text{Re} [\langle T(\mathbf{x}_1) T(\mathbf{x}_2) \rangle], \quad (4.6)$$

and

$$\begin{aligned} W^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= 2\text{Re} \left[\left\langle \frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_1)} \frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_2)} \frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_3)} - \left\{ \frac{\delta S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_1)} \frac{\delta^2 S_{\text{QFT}}}{\delta\zeta(\mathbf{x}_2) \delta\zeta(\mathbf{x}_3)} + (2 \text{ perms}) \right\} \right\rangle \Big|_{\zeta=0} \right] \\ &= 2\text{Re} \left[\left\langle T(\mathbf{x}_1) T(\mathbf{x}_2) T(\mathbf{x}_3) - (d-2) \{ T(\mathbf{x}_1) T(\mathbf{x}_2) \delta_\mu(\mathbf{x}_{23}) + (2 \text{ perms}) \} \right. \right. \\ &\quad \left. \left. - \{ T(\mathbf{x}_1) \partial T(\mathbf{x}_2) \delta_\mu(\mathbf{x}_{23}) + (2 \text{ perms}) \} \right\rangle \right]. \end{aligned} \quad (4.7)$$

Similarly, we can calculate the higher order vertex functions $W^{(n)}$ in terms of correlators of the energy momentum tensor. Here, we introduced

$$T(\mathbf{x}) \equiv \delta^{ij} T_{ij}(\mathbf{x})|_{\zeta=0}, \quad \partial T(\mathbf{x}) \equiv \frac{\partial T[\zeta](\mathbf{x})}{\partial\zeta(\mathbf{x})} \Big|_{\zeta=0}. \quad (4.8)$$

The coincidence limit is described by the smeared delta function $\delta_\mu(\mathbf{x})$ which takes a non-vanishing value only at $|\mathbf{x}| \leq 1/\mu$ and is normalized as

$$\int d^d \mathbf{x} \delta_\mu(\mathbf{x}) = 1. \quad (4.9)$$

Since the right hand side of Eq. (4.5) does depend on ζ , the n -th derivative with $n \geq 2$ of S_{QFT} does not vanish. Because of that, $W^{(n)}$ with $n \geq 3$ includes semi-local terms, where some of the arguments \mathbf{x}_i with $i = 1, \dots, n$ coincide, but the rest do not. The UV divergence from an ultra-local term, where all the arguments coincide, can be renormalized by using local counter terms. On the other hand, the regularization of the semi-local terms is not straightforward (see, e.g., Refs. [34, 37, 47]). Fortunately, for present purposes we will be able to sidestep this difficulty by using the consistency relations, as we shall see in the next Section.

Instead of using ζ , one may wish to introduce another variable $X(\mathbf{x})$, whose derivatives of W_t do not yield any semi-local terms [27, 38, 39]. For that, the boundary action S_{QFT} should depend on the new variable $X(\mathbf{x})$ only linearly. i.e.,

$$\frac{\delta^2 S_{\text{QFT}}}{\delta X(\mathbf{x}_1) \cdots \delta X(\mathbf{x}_n)} = 0 \quad (n \geq 2). \quad (4.10)$$

However, if X is a degree of freedom in the metric, it seems hard to find a variable that satisfies Eq. (4.10), because the action S_{QFT} non-linearly depends on the metric ⁵.

5. Conservation of ζ and consistency relation

In this section, we show that the WT identity for dilatation restricts the correlators of the energy-momentum tensor in the limit where some of the arguments coincide.

5.1 Cutoff independence of the energy momentum tensor

In Ref. [2], Callan, Coleman, and Jackiw (CCJ) considered the cutoff dependence of the so-called improved energy momentum tensor $\Theta_{\mu\nu}$. This differs from the conventional flat space symmetric energy momentum tensor by terms which are conserved identically⁶. CCJ showed that in a renormalizable theory, an insertion of $\Theta_{\mu\nu}$ to the correlators of the matter fields χ does not yield any cutoff dependence. Iterating the argument, it follows that n -successive insertions of Θ_{ij} do not give rise to any μ dependent contributions, as long as

⁵If the boundary action is given by a single trace operator O as

$$S_{\text{QFT}} = S_{\text{CFT}} + \int d\Omega \phi O,$$

we can choose $X(\mathbf{x})$ as $X(\mathbf{x}) = \delta\phi(\mathbf{x})$ (in the flat gauge) or a variable which is linearly related to $\delta\phi(\mathbf{x})$ such as $\zeta_n(\mathbf{x}) \equiv -(H/\dot{\phi})\delta\phi(\mathbf{x})$, which was introduced in Ref. [3]. Notice that since ζ and ζ_n are non-linearly related as presented in Eq. (A.8) of Ref. [3], $W^{(n)}$ with $n \geq 3$ for ζ include the semi-local terms as we discussed here.

⁶The improved energy momentum tensor can be obtained from the action of matter in curved space, with suitable non-minimal couplings to the metric, by taking functional derivative with respect to the metric and subsequently taking the flat space limit. It was shown in Ref. [48] that in order to establish the cut-off independence of the improved energy momentum tensor it is important to consider the running of the non-minimal coupling. More recently, this issue has been discussed in more detail, e.g., in Refs. [49, 50], following Ref. [51]. For our purposes, it will be sufficient to assume that the cut-off independent energy momentum tensor Θ_{ij} , can be obtained from the boundary theory by functional derivative with respect to the boundary metric, as in Eq. (4.4).

all the points are separated in position space. When some of the n -points coincide, the WT identity which was used in the discussion of CCJ potentially includes a momentum integral, which also integrates the UV modes and may induce a cutoff dependence.

In the following, we choose our energy momentum tensor for the boundary theory to be the improved energy-momentum tensor, *i.e.*, $T_{ij} = \Theta_{ij}$. In that case, we can express the correlators of T_{ij} (in the flat space limit) as

$$\text{Re} [\langle T_{ij}(\mathbf{x}_1) T_{kl}(\mathbf{x}_2) \rangle] = \text{Re} [\langle T_{ij}(\mathbf{x}_1) T_{kl}(\mathbf{x}_2) \rangle_0] , \quad (5.1)$$

$$\begin{aligned} \text{Re} [\langle T_{ij}(\mathbf{x}_1) T_{kl}(\mathbf{x}_2) T_{mn}(\mathbf{x}_3) \rangle] &= \text{Re} [\langle T_{ij}(\mathbf{x}_1) T_{kl}(\mathbf{x}_2) T_{mn}(\mathbf{x}_3) \rangle_0] \\ &+ \delta_\mu(\mathbf{x}_{12}) \mathcal{F}_{ij;kl;mn}(\mu; \mathbf{x}_{23}) + (2 \text{ perms}) , \end{aligned} \quad (5.2)$$

where $\langle T_{ij}(\mathbf{x}_1) T_{kl}(\mathbf{x}_2) \rangle_0$ and $\langle T_{ij}(\mathbf{x}_1) T_{kl}(\mathbf{x}_2) T_{mn}(\mathbf{x}_3) \rangle_0$ denote the μ independent contributions of the improved energy momentum tensor. CCJ's argument does not exclude the appearance of μ dependent contributions in the coincidence limit when two points coincide. The function $\mathcal{F}_{ij;kl;mn}(\mu; \mathbf{x}_{12})$ denotes the possible μ dependence from the coincidence limit. In Eqs. (5.1) and (5.2), we dropped the ultra-local terms, since these can be canceled by the local counterterms. Thus, the μ dependence can appear only in n -point correlators with $n > 2$.

In a bulk description, the local divergences correspond to a rapidly oscillating phase of the wave function Ψ , which cancels out in $|\Psi|^2$. In this paper, following Ref. [46], we will adopt the prescription (4.1), where the divergent phase contributions are canceled. (The phase contribution was briefly discussed in Ref. [3] and in more detail in Ref. [19].)

5.2 Consistency relation and coincidence limit

As shown in the previous section, the vertex functions $W^{(n)}$ are expressed in terms of the correlators of T and its derivative with respect to ζ . Using Eqs. (4.6) and (5.1), we obtain the Fourier mode of $W^{(2)}$ as

$$\delta(\mathbf{k}_1 + \mathbf{k}_2) \hat{W}^{(2)}(k_1) = -2 \prod_{i=1,2} \int d^d \mathbf{x}_i e^{i\mathbf{k}_i \cdot \mathbf{x}_i} \text{Re} [\langle T(\mathbf{x}_1) T(\mathbf{x}_2) \rangle_0] . \quad (5.3)$$

We find that the CCJ's argument directly implies that $\hat{W}^{(2)}(k)$ is μ independent or equivalently time independent in the bulk. When the perturbative expansion is possible, the μ independence of $\hat{W}^{(2)}(k)$ immediately leads to the μ independence of the power spectrum of ζ . In Ref. [1], assuming that the conformal symmetry is slightly broken by the deformation operator $\int d^d \Omega g \mathcal{O}$ in the boundary and solving the induced RG flow, the conservation of the power spectrum was explicitly shown. Here, we see that this result is much more general, and follows from the cut-off independence of the correlators of the energy-momentum tensor in a generic renormalizable boundary theory.

As we discussed in Sec. 3, the consistency relation (3.12) involves $W^{(n)}$ and $W^{(n-1)}$. Since $\hat{W}^{(2)}(k)$ is μ independent, Eq. (3.12) requires that $\hat{W}^{(3)}(0, k, k)$ should be also μ independent. Among the terms in $\hat{W}^{(3)}(0, k, k)$, a possible μ dependence can appear only from the terms with

$$\mathcal{F}(\mathbf{x}) \equiv \delta^{ij} \delta^{kl} \delta^{mn} \mathcal{F}_{ij;kl;mn}(\mathbf{x}) \quad (5.4)$$

and $\langle T\partial T \rangle$. Now, we find that the μ independence of $\hat{W}^{(3)}(0, k, k)$ requires ⁷

$$\frac{\partial}{\partial \mu} \Delta(\mu, k) = 0, \quad (5.5)$$

where we defined

$$\Delta(\mu, k) \equiv \hat{\mathcal{F}}(\mu; k) - \text{Re}[\langle T\partial T \rangle](\mu; k). \quad (5.6)$$

The condition (5.5) implies that the semi-local term $\langle T(\mathbf{x})\partial T(\mathbf{y}) \rangle$ should be related to the three-point function of T in the limit where two of the three arguments coincide, $\lim_{\mathbf{z} \rightarrow \mathbf{x}} \langle T(\mathbf{x})T(\mathbf{y})T(\mathbf{z}) \rangle$. The possible singular contribution which may arise in such coincidence limit should cancel out in the combination:

$$\text{Re} \left[\lim_{\mathbf{z} \rightarrow \mathbf{x}} \langle T(\mathbf{x})T(\mathbf{y})T(\mathbf{z}) \rangle - \langle \partial T(\mathbf{x})T(\mathbf{y}) \rangle \right].$$

It may seem surprising that a non-trivial condition is obtained from the requirement of the Diff invariance. This is because we are using ζ as our variable in the wave function, and this transforms under dilatation. If we could use a Diff invariant variable, we would not obtain any additional condition. In the bulk description, a technical difficulty to use such a Diff invariant variable for canonical quantization was pointed out in Refs. [52, 53, 54].

5.3 Conservation of ζ

In the previous subsection, we showed that CCJ's argument and the consistency relation imply the μ independence of $\hat{W}^{(3)}(0, k, k)$, which leads to the condition (5.5). Since $\Delta(k) \equiv \Delta(\mu, k)$ is μ independent, we may write

$$\begin{aligned} \hat{W}^{(3)}(k_1, k_2, k_3) &= 2\text{Re}[\langle TTT \rangle_0](k_1, k_2, k_3) \\ &\quad + 2 \sum_{i=1}^3 \{ \Delta(k_i) - (d-2)\text{Re}[\langle TT \rangle_0](k_i) \}, \end{aligned} \quad (5.7)$$

where $\text{Re}[\langle TT \rangle_0](k)$ and $\text{Re}[\langle TTT \rangle_0](k_1, k_2, k_3)$ denote the μ independent contributions in the Fourier modes of the real parts of the two and three point functions for T . In this way, the μ independence of $\hat{W}^{(3)}(k_1, k_2, k_{12})$ is established.

The μ independence of $\hat{W}^{(n)}$ for $n > 3$ can be derived recursively by a similar argument, starting with the assumption that $W^{(n-1)}$ is μ independent. The vertex functions $W^{(n)}$ are given by the correlators of T and $\delta^m S_{\text{QFT}}/\delta\zeta(\mathbf{x}_1) \cdots \delta\zeta(\mathbf{x}_m)|_{\zeta=0}$ with $m < n$. The correlators with $\delta^m S_{\text{QFT}}/\delta\zeta(\mathbf{x}_1) \cdots \delta\zeta(\mathbf{x}_m)|_{\zeta=0}$ contribute to the coincidence limit where $\mathbf{x}_1, \dots, \mathbf{x}_m$ agree, and they can depend on μ . Meanwhile, in this coincidence limit, the auto-correlation functions of T also can depend on μ . The consistency relation then requires

⁷What we directly obtain from the μ independence of $\hat{W}^{(3)}(0, k, k)$ is

$$\frac{\partial}{\partial \mu} [\Delta(\mu, k=0) + 2\Delta(\mu, k)] = 0.$$

First, we set $k = 0$, then we find that $\Delta(\mu, 0)$ is μ independent, which implies $\Delta(\mu, k)$ with $k \neq 0$ is also μ independent.

that $W^{(n)}$ with one or more of the momenta set to 0 should be μ independent. This determines a relation between all μ dependent contributions, so that they cancel out in the vertex with a soft leg. From that relation, it can be shown that the vertex function $W^{(n)}$ without any soft leg is also μ independent. Finally, the μ independence of all $W^{(n)}$ ensures that the probability distribution $P[\zeta] = e^{-W[\zeta]}$ (and therefore all correlators of ζ) are “time” independent, once we interpret the RG flow as time evolution.

We noted in Subsection 3.3 that, in order to rewrite the consistency relation in terms of ζ correlators at the tree level, in the form Eq. (3.15), it is necessary to assume the continuity of the vertex function $W^{(n)}$ at $\mathbf{k} = 0$. In the holographic context, this is equivalent to the continuity of the energy momentum tensor (and its derivative) at $\mathbf{k} = 0$.

Finally, let us note that in Ref. [2] the infrared behaviour of correlators with an insertion of T_{ij} was assumed to be analytic. In this situation, the vertex functions of the boundary theory are also expected to be analytic in the infrared, and the μ independence of $W^{(n)}(\{\mathbf{k}_i\}_n)$ follows directly from the μ independence of $W^{(n)}(\mathbf{k} = 0, \{\mathbf{k}_i\}_{n-1})$. In that case the correction for a finite \mathbf{k} will be suppressed by $(k/\mu)^p$ with an integer positive power of p , and will vanish in the long wavelength limit. Notice that, in the present holographic context, where the CCJ argument can be applied, a possibility that ζ has a growing mode even in one field models (see, e.g., Ref. [55]) is excluded.

5.4 Primordial spectra from holographic inflation

When the amplitude of ζ is sufficiently small, we can compute the spectra of ζ using the formulae derived in Section 2.2. Inserting $\hat{W}^{(2)}$ and $\hat{W}^{(3)}$ into Eqs. (2.11) and (2.14), we obtain the power spectrum and the bi-spectrum for ζ as

$$P(k) = -\frac{1}{2\text{Re}[\langle TT \rangle_0(k)]}, \quad (5.8)$$

$$B(k_1, k_2, k_3) = \frac{1}{4} \frac{1}{\prod_{i=1}^3 \text{Re}[\langle TT \rangle_0(k_i)]} \left[\text{Re}[\langle TTT \rangle_0](k_1, k_2, k_3) - \sum_{i=1}^3 \{(d-2)\text{Re}[\langle TT \rangle_0](k_i) - \Delta(k_i)\} \right], \quad (5.9)$$

where k_i should satisfy $\sum_{i=1}^3 \mathbf{k}_i = 0$. This formulae should hold generically for a holographic model which is dual to a single clock inflation, as far as the diffeomorphism invariance in the boundary is preserved and the boundary theory is renormalizable. To compute the bi-spectrum of ζ , we need to know the correlators in the coincidence limit, described by $\Delta(k)$.

5.5 Relation to previous work

In Ref. [1], assuming that the boundary theory is given by a single trace operator \mathcal{O} as

$$S_{\text{QFT}} = S_{\text{CFT}} + \int d^d\Omega \phi \mathcal{O}. \quad (5.10)$$

we addressed the conservation of ζ . The second term describes the deformation from the CFT, which is needed to be dual to an inflationary spacetime. The (dimensionless) coupling

constant ϕ plays the role of the inflaton. By solving the RG flow, it was shown that when all the arguments \mathbf{x}_i with $i = 1, \dots, n$ are separated by a distance larger than $1/\mu$, the correlators of \mathcal{O} satisfy

$$Z^{-n/2}(\mu)\langle\mathcal{O}(\mathbf{x}_1)\cdots\mathcal{O}(\mathbf{x}_n)\rangle_\mu = Z^{-n/2}(\mu_0)\langle\mathcal{O}(\mathbf{x}_1)\cdots\mathcal{O}(\mathbf{x}_n)\rangle_{\mu_0}, \quad (5.11)$$

where $Z(\mu)$ denotes the wave function renormalization. It follows very simply from this relation that $\hat{W}^{(2)}$ is μ independent.

In Ref. [1], it was assumed that the n -point function of \mathcal{O} for an arbitrary configuration (including the coincidence limit, where some of the arguments are closer than $1/\mu$) is given by Eq. (5.11). Under this assumption, it was shown that $\hat{W}^{(3)}$ cannot be μ independent, except for a particular case where the beta function $\beta \equiv d\phi/d\ln\mu$ is given by a simple power law scaling as $\beta \propto \mu^\lambda$ with a constant parameter λ . Thus, the conservation of $W^{(n)}$ for $n \geq 3$ in a more generic RG flow is not compatible with this assumption. On the other hand this assumption was not particularly well motivated, since there is no reason to expect that the expression (5.11) can be extrapolated to the coincidence limit. As we have shown in this section, the μ dependent contribution should exist in the coincidence limit, in order to cancel the μ dependent semi-local contribution, so that the consistency relation (3.12) can be satisfied.

Another possibly relevant issue is that the setup given by Eq. (5.10) might be too simplistic. In Refs. [56, 57], a Wilsonian treatment of the holographic RG flow was explored in the context of AdS/CFT. According to their bulk computations, integrating out the UV contributions generates multi trace operators in the boundary theory (the RG flow with multi trace operators in dS/CFT setup was discussed in Ref. [58]). In Refs. [59, 60], starting from a boundary theory with multi trace operators, it was shown that the Einstein gravity emerges after a particular mapping between the boundary and the bulk quantities. Motivated by these studies, one may generalize the boundary QFT to a more general theory which includes both single and multi trace operators as

$$S_{\text{QFT}}[\phi, \mathcal{O}] = S_{\text{CFT}} + \int d^d\Omega \phi \mathcal{O} + \int d^d\Omega \mathcal{F}[\phi] \mathcal{O} \partial \mathcal{O} + \dots, \quad (5.12)$$

where ∂ is a derivative operator, and the ellipsis denote possible multi trace operators with more \mathcal{O} . The coefficients ϕ and $\mathcal{F}[\phi]$ will depend on the bulk gravity theory. It might be interesting to see if the consistency relation (3.12) imposes any meaningful restriction on the multi-trace operators or not. We leave this for a future study.

6. Condition on boundary theory from diffeomorphism invariance

So far, we have considered only the WT identity for the dilatation, which restricts the vertex function for the scalar perturbation and, in turn, the trace part of the energy-momentum tensor in the boundary. As was pointed out in Refs. [52, 53] (see also Ref. [61]), the gauge:

$$\delta\phi = 0, \quad (6.1)$$

and

$$dl_d^2 = h_{ij} dx^i dx^j = a^2 e^{2\zeta} [e^\gamma]_{ij} dx^i dx^j \quad (6.2)$$

with

$$\partial_i \gamma_{ij} = \gamma_{ii} = 0, \quad (6.3)$$

accommodates an infinite number of residual gauge degrees of freedom (in the choice of the spatial coordinates), when we are concerned only with a causally connected region to us. These residual gauge degrees of freedom describe the variation of the boundary conditions imposed at the null boundary of the causally connected region ⁸.

Among these residual gauge degrees of freedom, we consider the transformation

$$x^i \rightarrow \tilde{x}^i \equiv \Lambda_j^i x^j \equiv (\delta_j^i + \delta \Lambda_j^i) x^j \quad (6.4)$$

with

$$\Lambda_j^i \equiv e^s [e^S]_j^i, \quad (6.5)$$

which additionally includes a constant symmetric tensor S_{ij} . Under this transformation, the above gauge condition remains unchanged. The diffeomorphism invariance of the wave function ψ_t requires that W_t should remain unchanged under this residual gauge transformation as

$$W_t[h_{ij}(\mathbf{x})] = W_t[\tilde{h}_{ij}(\mathbf{x})], \quad (6.6)$$

where \tilde{h}_{ij} is related to h_{ij} as

$$\tilde{h}_{ij}(\tilde{\mathbf{x}}) = (\Lambda^{-1})_i^k (\Lambda^{-1})_j^l h_{kl}(\mathbf{x}). \quad (6.7)$$

Here, Λ^{-1} denotes the inverse matrix of Λ . Repeating a similar argument, Eq. (6.6) gives the WT identity

$$0 = \delta \Lambda_l^k \int d^d \mathbf{x} \left[\frac{\delta W_t}{\delta h_{ij}} x^l \partial_k h_{ij} + 2 \frac{\delta W_t}{\delta h_{il}} h_{ki} \right]. \quad (6.8)$$

This is a generalization of Eq. (3.6).

As one may expect, the WT identity (6.8) is nothing but the conservation of the energy momentum tensor on the boundary. In fact, we can show that Eq. (6.8) is equivalent to

$$0 = \int d^d \mathbf{x} \sqrt{h} h_{il} \delta x^l \nabla_j \text{Re}[\langle T^{ij} \rangle] \quad (6.9)$$

with $\delta x^i \equiv \tilde{x}^i - x^i$. Here, we used

$$\frac{\delta W_t}{\delta h_{ij}} = 2 \text{Re} \left[\frac{\delta W_{\text{QFT}}}{\delta h_{ij}} \right] = -\sqrt{h} \text{Re}[\langle T^{ij} \rangle]. \quad (6.10)$$

⁸It was shown that these residual gauge degrees of freedom give rise to the infrared divergence of ζ and γ_{ij} (see, e.g., Refs. [54, 62, 63, 64]).

Since this is derived from the diffeomorphism invariance of W_t , we obtain only the real part of the conservation of the energy momentum tensor. Meanwhile, the diffeomorphism invariance of W_{QFT} gives $\nabla_i \langle T^{ij} \rangle = 0$ (see also Ref. [44]). In the gauge with $\delta\phi = 0$, because of the absence of the external source other than the metric, the expectation value of the energy-momentum tensor is conserved unlike in the Fefferman and Graham gauge, where the energy-momentum tensor is not conserved due to the presence of the external source scalar field [65].

Repeating a similar argument to the one in the previous section, with the use of the WT identity (6.8), we can derive the consistency relation for the scalar and tensor perturbations. (Notice that the Diff invariance plays the crucial role both in the consistency relation and the CCJ.) Similarly, the consistency relation gives the conditions on the coincidence limit, e.g., the trace part and the transverse and traceless part of

$$\text{Re} \left[\lim_{z \rightarrow x} \langle T_{ij}(\mathbf{x}) T_{kl}(\mathbf{y}) T_{mn}(\mathbf{z}) \rangle - 2 \left\langle \frac{\partial T_{ij}(\mathbf{x})}{\partial h_{mn}(\mathbf{x})} T_{kl}(\mathbf{y}) \right\rangle \right] \quad (6.11)$$

should not give rise to singular contributions.

7. Conclusions

The wave function provides a characterization of the distribution function $P_t[\zeta] = |\psi_t[\zeta]|^2 \propto e^{-W_t[\zeta]}$ of primordial fluctuations in the curvature perturbation ζ . This characterization does not require the amplitudes of the fluctuations to be perturbatively small, and can be applied non-perturbatively. In the first half of this paper, we derived the consistency relation for the vertex functions $W^{(n)}$, which are the coefficients of the expansion of W_t in powers of $\zeta(\mathbf{x})$. The consistency relation is derived by using the Ward-Takahashi identity associated with dilatation invariance of the wave function, and provides a generalization of the well-known consistency relation for the tree level correlation functions of ζ , which relates the n -point function with one soft leg to the $(n-1)$ -point function.

In the second part of the paper, postulating the holographic duality between the $(d+1)$ -dimensional cosmological spacetime and the d -dimensional boundary theory, we discussed the conservation of the curvature perturbation. The main ingredients we used in our discussion are the cut-off independence of correlators of the energy momentum tensor and the consistency relations amongst vertex functions. The first assumption is known to be valid in renormalizable field theories, as long as we use the appropriately “improved” energy momentum tensor. The second ingredient is simply a consequence of Diff invariance. First, we showed that the consistency relation provides a condition on the semi-local terms, where some of the arguments coincide. When this condition is fulfilled, we can show the cutoff independence of the vertex function $W^{(n)}$. In the bulk perspective, this implies the conservation of ζ in time at large scales.

An intriguing feature of the holographic approach is that the conservation of $W^{(2)}$, and hence of the tree level power spectrum for ζ , follows simply from the cut-off independence of the correlator of T . In a bulk description, the long wavelength solution for ζ can have a growing mode even in one field models, when these exhibit non-attractor behaviour.

This possibility seems to be excluded in the present holographic context, where the CCJ argument applies. A possible way around that is to consider a boundary theory where the analyticity in the IR is violated. In that case we may expect μ dependence of correlators of ζ . Conversely, from a bulk perspective, non-analyticity in the IR is expected in the case when the long wavelength ζ is not constant in time. The reason is that the shift vector N_i is related to ζ as $\partial^i N_i \propto \dot{\zeta}$, which introduces a non-local contribution with a negative power of k when we eliminate N_i by using this relation.

Acknowledgments

We are grateful to David Mateos, Yu Nakayama, Tadakatsu Sakai, Sergey Sibiryakov, and Takahiro Tanaka for their valuable comments. Y. U. would like to thank Institute for Advanced Study and Tufts university for their warm hospitalities during which a part of this project was proceeded. J. G. is supported by MEC FPA2013-46570-C2-2-P, AGAUR 2014-SGR- 1474, and CPAN CSD2007-00042 Consolider-Ingenio 2010. Y. U. is supported by JSPS Grant-in-Aid for Research Activity Start-up under Contract No. 26887018 and Grant-in-Aid for Scientific Research on Innovative Areas under Contract No. 16H01095. This research was supported in part by Building of Consortia for the Development of Human Resources in Science and Technology and the National Science Foundation under Grant No. NSF PHY11-25915.

A. Consistency relations for ζ correlators

Using Eq. (3.12), we can derive the well-known consistency relation for the curvature perturbation ζ . Using Eq. (3.12) for $n = 3$ divided by the square of $\hat{W}^{(2)}(t; k_2)$, we obtain

$$\frac{\hat{W}^{(3)}(t; 0, k_2, k_2)}{\{\hat{W}^{(2)}(t; k_2)\}^2} - (\mathbf{k}_2 \cdot \partial_{\mathbf{k}_2} + d) \frac{1}{\hat{W}^{(2)}(t, k_2)} = 0. \quad (\text{A.1})$$

Since the power spectrum and the bispectrum are given by Eqs. (2.11) and (2.14), respectively, we find that Eq. (A.1) indeed gives the consistency relation for the bi-spectrum:

$$\lim_{k_3/k_1, k_3/k_2 \rightarrow 0} \frac{B(t; k_1, k_2, k_3)}{P(k_3)} = -(\mathbf{k}_2 \cdot \partial_{\mathbf{k}_2} + d)P(k_2). \quad (\text{A.2})$$

A systematic derivation can be given by using the mathematical induction. Dividing Eq. (3.12) by

$$\prod_{i=1}^{n-1} \hat{W}^{(2)}(k_i) = \hat{W}^{(2)}(K_{n-1}) \hat{W}^{(2)}(k_2) \cdots \hat{W}^{(2)}(k_{n-1}), \quad (\text{A.3})$$

we obtain

$$\begin{aligned} & -\frac{\hat{W}^{(n)}(0, \{\mathbf{k}_i\}_{n-1})}{\prod_{i=1}^{n-1} \hat{W}^{(2)}(k_i)} + \left(\sum_{i=1}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} \ln \frac{k_i^d}{\hat{W}^{(2)}(k_i)} \right) \frac{\hat{W}^{(n-1)}(\{\mathbf{k}_i\}_{n-1})}{\prod_{i=1}^{n-1} \hat{W}^{(2)}(k_i)} \\ & = - \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} + d(n-2) \right) \left\{ -\frac{\hat{W}^{(n-1)}(\{\mathbf{k}_i\}_{n-1})}{\prod_{i=1}^{n-1} \hat{W}^{(2)}(k_i)} \right\}, \end{aligned} \quad (\text{A.4})$$

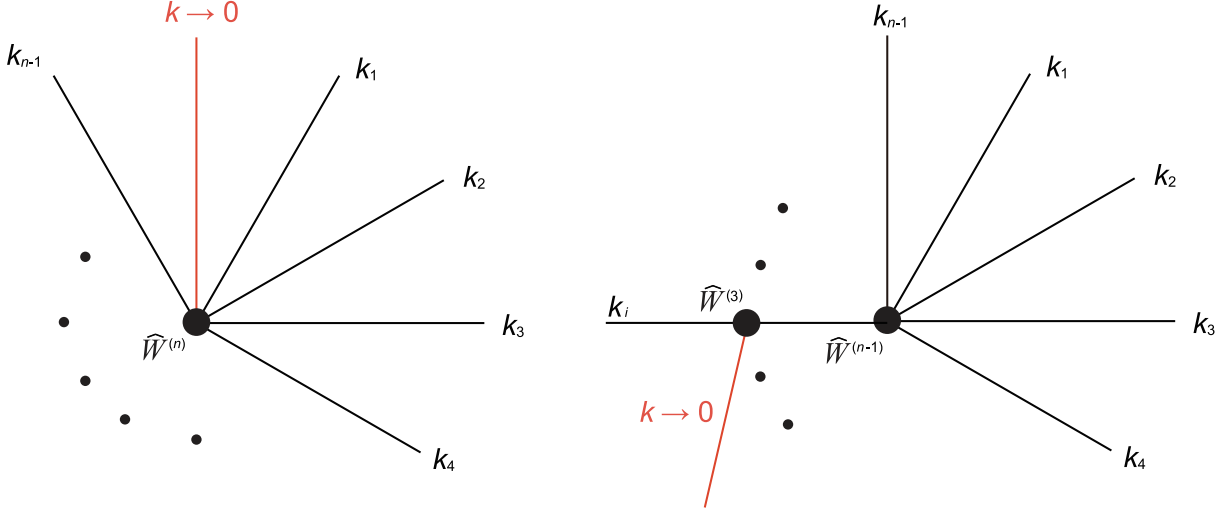


Figure 1: The left diagram is the Feynman diagram for the first term in the first line of Eq. (A.4) and the right diagram is the one for the second one with the summation index i .

where we used

$$\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} \hat{W}^{(2)}(K_{n-1}) = \mathbf{K}_{n-1} \cdot \partial_{\mathbf{K}_{n-1}} \hat{W}^{(2)}(K_{n-1}). \quad (\text{A.5})$$

The first term in the first line of Eq. (A.4) is the contribution from the left diagram of Fig. 1 with one of the n -momenta sent to 0. The minus sign appears, since the n -point interaction is given by $-\hat{W}^{(n)}(\{\mathbf{k}_i\}_n)$. Since the momentum derivative in the second term can be rewritten as

$$\mathbf{k}_i \cdot \partial_{\mathbf{k}_i} \ln \frac{k_i^d}{\hat{W}^{(2)}(k_i)} = -f_{NL}^{local}(k_i) = \frac{\hat{W}^{(3)}(0, k_i, k_i)}{\hat{W}^{(2)}(k_i)}, \quad (\text{A.6})$$

we find that the second term is the contribution from the right diagram of Fig. 1.

The expression in the braces in the second line denotes the contribution from the one particle irreducible diagram for the $(n-1)$ -point function, which cannot be decomposed into two diagrams by cutting one of the propagators included in the diagram. For $n=4$, Eq. (A.4) gives the consistency relation (3.15), since the three point function contains only the irreducible diagram. For $n > 4$, the $(n-1)$ -point function also contains reducible diagrams and hence we obtain

$$\begin{aligned} & -\frac{\hat{W}^{(n)}(0, \{\mathbf{k}_i\}_{n-1})}{\prod_{i=1}^{n-1} \hat{W}^{(2)}(k_i)} + \left(\sum_{i=1}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} \ln \frac{k_i^d}{\hat{W}^{(2)}(k_i)} \right) \frac{\hat{W}^{(n-1)}(\{\mathbf{k}_i\}_{n-1})}{\prod_{i=1}^{n-1} \hat{W}^{(2)}(k_i)} \\ & - \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} + d(n-2) \right) (\text{Contributions from reducible diagrams}) \\ & = - \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} + d(n-2) \right) \mathcal{C}^{(n-1)}(\{\mathbf{k}_i\}_{n-1}). \end{aligned} \quad (\text{A.7})$$

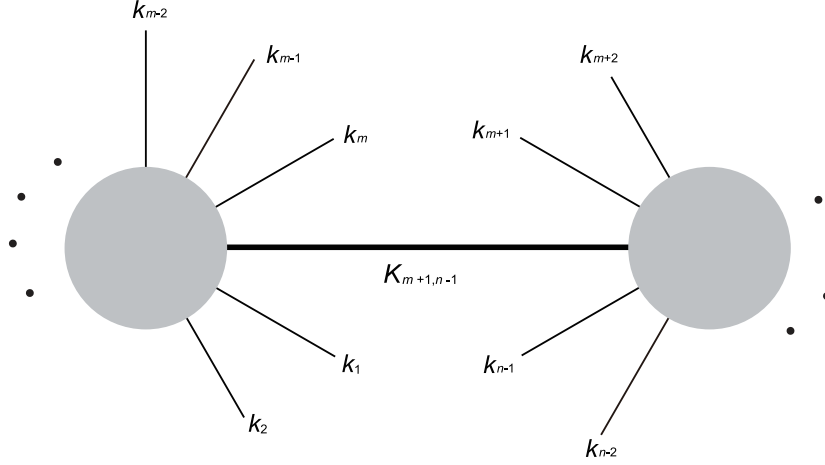


Figure 2: A 1PI reducible diagram which shows up in the second line of Eq. (A.7). We highlight the double counted propagator by the thick line.

Now, all we need to show is that the summation of the first two lines of Eq. (A.7) gives the left-hand side of the consistency relation (3.15). In order to compute the terms in the second line, we consider a 1PI reducible diagram which is depicted in Fig. 2. This diagram can be understood as connecting the two diagrams, the $(m+1)$ -point function with momenta $\{\mathbf{k}_i\}_m$ and $\mathbf{K}_{m+1, n-1} = \sum_{i=m+1}^{n-1} \mathbf{k}_i$ and $(n-m)$ -point function with momenta $-\mathbf{K}_{m+1, n-1}$ and \mathbf{k}_j for $j = m+1, \dots, n-1$. These two diagrams are joined by the propagator with the momentum $\mathbf{K}_{m+1, n-1}$. Here m is $2 \leq m \leq n-3$. This contribution is given by

$$\begin{aligned} & \mathcal{C}^{(m+1)}(\mathbf{K}_{2, n-1}, \mathbf{k}_2, \dots, \mathbf{k}_m, \mathbf{K}_{m+1, n-1}) \\ & \times \mathcal{C}^{(n-m)}(\mathbf{K}_{m+1, n-1}, \mathbf{k}_{m+1}, \dots, \mathbf{k}_{n-1}) \hat{W}^{(2)}(K_{m+1, n-1}), \end{aligned} \quad (\text{A.8})$$

where the propagator $P(K_{m+1, n-1}) = 1/\hat{W}^{(2)}(K_{m+1, n-1})$ is counted twice in the jointed two diagrams, so we divided by it. Operating $\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i}$ on Eq. (A.8) and using the consistency relation (3.15) for the $(m+2)$ -point and $(n-m+1)$ -point functions in the squeezed limit, we obtain

$$\begin{aligned} & - \left(\sum_{i=2}^{n-1} \mathbf{k}_i \cdot \partial_{\mathbf{k}_i} + d(n-2) \right) (\text{Reducible diagram in Fig. 1}) \\ & = \lim_{\mathbf{k} \rightarrow 0} \frac{\mathcal{C}^{(m+2)}(\mathbf{k}, \{\mathbf{k}_i\}_m, \mathbf{K}_{m+1, n-1}) \mathcal{C}^{(n-m)}(\mathbf{K}_{m+1, n-1}, \mathbf{k}_{m+1}, \dots, \mathbf{k}_{n-1}) \hat{W}^{(2)}(K_{m+1, n-1})}{P(k)} \\ & + \lim_{\mathbf{k} \rightarrow 0} \frac{\mathcal{C}^{(m+1)}(\{\mathbf{k}_i\}_m, \mathbf{K}_{m+1, n-1}) \mathcal{C}^{(n-m+1)}(\mathbf{k}, \mathbf{K}_{m+1, n-1}, \mathbf{k}_{m+1}, \dots, \mathbf{k}_{n-1}) \hat{W}^{(2)}(K_{m+1, n-1})}{P(k)} \\ & - \lim_{\mathbf{k} \rightarrow 0} \frac{1}{P(k)} \left[\mathcal{C}^{(m+1)}(\{\mathbf{k}_i\}_m, \mathbf{K}_{m+1, n-1}) \left(- \frac{\hat{W}^{(3)}(k, K_{m+1, n-1}, K_{m+1, n-1})}{\hat{W}^{(2)}(k)} \right) \right. \\ & \quad \left. \times \mathcal{C}^{(n-m)}(\mathbf{K}_{m+1, n-1}, \mathbf{k}_{m+1}, \dots, \mathbf{k}_{n-1}) \right]. \end{aligned} \quad (\text{A.9})$$

The term in the second line is the contribution from the diagram with one soft leg inserted to the $(m+1)$ -point function and the one in the third line is from the diagram with one soft leg inserted to the $(n-m)$ -point function. Since the diagram with one soft leg inserted to the joint propagator with the momentum $\mathbf{K}_{m+1, n-1}$ is counted both in the second and third lines and hence the term in the last two lines cancels the doubled counted one. In this way, we can see that the terms in the first two lines of Eq. (A.7) add up the contributions from all the diagrams for $\mathcal{C}^{(n)}(\mathbf{k}, \{\mathbf{k}_i\}_{n-1})$ in the squeezed limit, $\mathbf{k} \rightarrow 0$. Now, the consistency relation for the n -point function, which was derived in Ref. [10], is extended to an arbitrary spacetime dimension, following a different method.

References

- [1] J. Garriga and Y. Urakawa, JHEP **1406**, 086 (2014) [arXiv:1403.5497 [hep-th]].
- [2] C. G. Callan, Jr., S. R. Coleman and R. Jackiw, Annals Phys. **59**, 42 (1970).
doi:10.1016/0003-4916(70)90394-5
- [3] J. M. Maldacena, JHEP **0305**, 013 (2003) [astro-ph/0210603].
- [4] J. Garriga and Y. Urakawa, JCAP **1307**, 033 (2013) [arXiv:1303.5997 [hep-th]].
- [5] W. D. Goldberger, L. Hui and A. Nicolis, Phys. Rev. D **87**, no. 10, 103520 (2013)
doi:10.1103/PhysRevD.87.103520 [arXiv:1303.1193 [hep-th]].
- [6] L. Berezhiani and J. Khoury, JCAP **1402**, 003 (2014) doi:10.1088/1475-7516/2014/02/003
[arXiv:1309.4461 [hep-th]].
- [7] J. A. Harvey, In *Trieste 1995, High energy physics and cosmology* 66-125 [hep-th/9603086].
- [8] S. G. Avery and B. U. W. Schwab, JHEP **1602**, 031 (2016) doi:10.1007/JHEP02(2016)031
[arXiv:1510.07038 [hep-th]].
- [9] M. Mirbabayi and M. Simonovic, arXiv:1602.05196 [hep-th].
- [10] K. Hinterbichler, L. Hui and J. Khoury, JCAP **1401**, 039 (2014) [arXiv:1304.5527 [hep-th]].
- [11] T. Tanaka and Y. Urakawa, JCAP **1105**, 014 (2011) doi:10.1088/1475-7516/2011/05/014
[arXiv:1103.1251 [astro-ph.CO]].
- [12] E. Pajer, F. Schmidt and M. Zaldarriaga, Phys. Rev. D **88**, no. 8, 083502 (2013)
doi:10.1103/PhysRevD.88.083502 [arXiv:1305.0824 [astro-ph.CO]].
- [13] T. Tanaka and Y. Urakawa, arXiv:1510.05059 [hep-th].
- [14] S. Weinberg, Phys. Rev. D **67**, 123504 (2003) doi:10.1103/PhysRevD.67.123504
[astro-ph/0302326].
- [15] A. Strominger, JHEP **0110**, 034 (2001) [hep-th/0106113].
- [16] A. Strominger, JHEP **0111**, 049 (2001) [hep-th/0110087].
- [17] E. Witten, hep-th/0106109.
- [18] R. Bousso, A. Maloney and A. Strominger, Phys. Rev. D **65**, 104039 (2002) [hep-th/0112218].
- [19] D. Harlow and D. Stanford, arXiv:1104.2621 [hep-th].
- [20] D. Anninos, T. Hartman and A. Strominger, arXiv:1108.5735 [hep-th].

- [21] F. Larsen and R. McNees, JHEP **0307**, 051 (2003) [hep-th/0307026].
- [22] F. Larsen and R. McNees, JHEP **0407**, 062 (2004) [hep-th/0402050].
- [23] D. Seery and J. E. Lidsey, JCAP **0606**, 001 (2006) [astro-ph/0604209].
- [24] J. P. van der Schaar, JHEP **0401**, 070 (2004) [hep-th/0307271].
- [25] K. Schalm, G. Shiu and T. van der Aalst, arXiv:1211.2157 [hep-th].
- [26] I. Mata, S. Raju and S. Trivedi, JHEP **1307**, 015 (2013) [arXiv:1211.5482 [hep-th]].
- [27] A. Ghosh, N. Kundu, S. Raju and S. P. Trivedi, arXiv:1401.1426 [hep-th].
- [28] T. Banks, W. Fischler, T. J. Torres and C. L. Wainwright, arXiv:1306.3999 [hep-th].
- [29] T. Banks, arXiv:1311.0755 [hep-th].
- [30] G. L. Pimentel, JHEP **1402**, 124 (2014) [arXiv:1309.1793 [hep-th]].
- [31] P. McFadden and K. Skenderis, Phys. Rev. D **81**, 021301 (2010) [arXiv:0907.5542 [hep-th]].
- [32] P. McFadden, K. Skenderis, J. Phys. Conf. Ser. **222**, 012007 (2010). [arXiv:1001.2007 [hep-th]].
- [33] P. McFadden, K. Skenderis, [arXiv:1010.0244 [hep-th]].
- [34] P. McFadden, K. Skenderis, JCAP **1105**, 013 (2011). [arXiv:1011.0452 [hep-th]].
- [35] P. McFadden and K. Skenderis, JCAP **1106**, 030 (2011) [arXiv:1104.3894 [hep-th]].
- [36] E. Kiritsis, JCAP **1311**, 011 (2013) [arXiv:1307.5873 [hep-th]].
- [37] A. Bzowski, P. McFadden and K. Skenderis, JHEP **1304**, 047 (2013) [arXiv:1211.4550 [hep-th]].
- [38] N. Kundu, A. Shukla and S. P. Trivedi, JHEP **1504**, 061 (2015) doi:10.1007/JHEP04(2015)061 [arXiv:1410.2606 [hep-th]].
- [39] N. Kundu, A. Shukla and S. P. Trivedi, arXiv:1507.06017 [hep-th].
- [40] J. Garriga and A. Vilenkin, JCAP **0911**, 020 (2009) doi:10.1088/1475-7516/2009/11/020 [arXiv:0905.1509 [hep-th]].
- [41] J. Garriga, K. Skenderis and Y. Urakawa, JCAP **1501**, no. 01, 028 (2015) doi:10.1088/1475-7516/2015/01/028 [arXiv:1410.3290 [hep-th]].
- [42] J. Garriga, Y. Urakawa and F. Vernizzi, JCAP **1602**, no. 02, 036 (2016) doi:10.1088/1475-7516/2016/02/036 [arXiv:1509.07339 [hep-th]].
- [43] P. McFadden, JHEP **1310**, 071 (2013) doi:10.1007/JHEP10(2013)071 [arXiv:1308.0331 [hep-th]].
- [44] P. McFadden, JHEP **1502**, 053 (2015) doi:10.1007/JHEP02(2015)053 [arXiv:1412.1874 [hep-th]].
- [45] S. Kawai and Y. Nakayama, JHEP **1406**, 052 (2014) doi:10.1007/JHEP06(2014)052 [arXiv:1403.6220 [hep-th]].
- [46] D. Harlow and L. Susskind, arXiv:1012.5302 [hep-th].
- [47] A. Bzowski and K. Skenderis, JHEP **1408**, 027 (2014) doi:10.1007/JHEP08(2014)027 [arXiv:1402.3208 [hep-th]].

- [48] J. C. Collins, Phys. Rev. D **14**, 1976, p. 1965-1976
- [49] K. Yonekura, JHEP **1304**, 011 (2013) doi:10.1007/JHEP04(2013)011 [arXiv:1212.3028 [hep-th]].
- [50] Y. Nakayama, Phys. Rept. **569**, 1 (2015) doi:10.1016/j.physrep.2014.12.003 [arXiv:1302.0884 [hep-th]].
- [51] J. Polchinski, Nucl. Phys. B **303** (1988) 226. doi:10.1016/0550-3213(88)90179-4
- [52] Y. Urakawa and T. Tanaka, Phys. Rev. D **82**, 121301 (2010) doi:10.1103/PhysRevD.82.121301 [arXiv:1007.0468 [hep-th]].
- [53] Y. Urakawa and T. Tanaka, Prog. Theor. Phys. **125**, 1067 (2011) doi:10.1143/PTP.125.1067 [arXiv:1009.2947 [hep-th]].
- [54] T. Tanaka and Y. Urakawa, Class. Quant. Grav. **30**, 233001 (2013) doi:10.1088/0264-9381/30/23/233001 [arXiv:1306.4461 [hep-th]].
- [55] M. H. Namjoo, H. Firouzjahi and M. Sasaki, Europhys. Lett. **101**, 39001 (2013) doi:10.1209/0295-5075/101/39001 [arXiv:1210.3692 [astro-ph.CO]].
- [56] I. Heemskerk and J. Polchinski, JHEP **1106**, 031 (2011) [arXiv:1010.1264 [hep-th]].
- [57] T. Faulkner, H. Liu and M. Rangamani, JHEP **1108**, 051 (2011) [arXiv:1010.4036 [hep-th]].
- [58] D. Das, S. R. Das and G. Mandal, JHEP **1311**, 186 (2013) doi:10.1007/JHEP11(2013)186 [arXiv:1306.0336 [hep-th]].
- [59] S. S. Lee, JHEP **1210**, 160 (2012) [arXiv:1204.1780 [hep-th]].
- [60] S. S. Lee, JHEP **1401**, 076 (2014) [arXiv:1305.3908 [hep-th]].
- [61] K. Hinterbichler, L. Hui and J. Khoury, JCAP **1208**, 017 (2012) [arXiv:1203.6351 [hep-th]].
- [62] T. Tanaka and Y. Urakawa, PTEP **2013**, 083E01 (2013) doi:10.1093/ptep/ptt057 [arXiv:1209.1914 [hep-th]].
- [63] T. Tanaka and Y. Urakawa, PTEP **2013**, no. 6, 063E02 (2013) doi:10.1093/ptep/ptt037 [arXiv:1301.3088 [hep-th]].
- [64] T. Tanaka and Y. Urakawa, PTEP **2014**, no. 7, 073E01 (2014) doi:10.1093/ptep/ptu071 [arXiv:1402.2076 [hep-th]].
- [65] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. **217**, 595 (2001) doi:10.1007/s002200100381 [hep-th/0002230].